

Transformation of the Dirac-Born-Infeld action under the Seiberg-Witten map

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ABSTRACT

We explicitly evaluate the disk S-matrix elements of one closed string and an arbitrary number of open string states in the presence of a large background B-flux. From this calculation, we show that in the world-volume action of D-branes in terms of non-commutative fields, the closed string fields must be treated as functionals of the non-commutative gauge fields. We also find the generalized multiplication rule $*_N$ between N open string fields on the world-volume of the D-brane. In particular, this result indicates that the difference between the familiar $*$ and the $*_N$ product is just some total derivative terms. We show that the $*_N$ product and the dependence of the closed string fields on the non-commutative gauge fields emerge also from transforming the ordinary Dirac-Born-Infeld action (including the closed string fields) under the Seiberg-Witten map. We then conjecture a non-commutative DBI action for the transformation of the commutative DBI action under the SW map.

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1 Introduction

Recent years have seem dramatic progress in the understanding of non-perturbative aspects of string theory – see, *e.g.*, [1]. With these studies has come the realization that solitonic extended objects, other than just strings, play an essential role. An important object in these investigations has been Dirichlet branes [2]. D-branes are non-perturbative states whose perturbative excitations are described by the fundamental open string states on their world-volumes. They also couple to various perturbative closed string states including the Ramond-Ramond states.

The low energy effective world-volume theory of a single D-brane is a $U(1)$ gauge theory. An interesting extension of a gauge theory is its generalization to non-commutative theory. Recently, it has been shown that certain non-commutative gauge theories emerge naturally as the D-brane world-volume theory when a constant NS-NS background B-flux is turned on [3, 4, 5, 6, 7, 8, 9]. Hence, one may study the D-brane with the B-flux to learn more about the non-commutative gauge theories. An aspect of non-commutative gauge theories that the D-brane study reveals is that the multiplication rule between gauge fields in general is not just the $*$ product[10]. The new multiplication rule $*'$ between two gauge fields appears when one studies the S-matrix elements of one closed and two open string states at tree level. This multiplication rule appears also in calculating the S-matrix element of four open string states at one loop level[11, 12, 13], and in studying in detail anomalies in non-commutative gauge theories[14]. In [11], another multiplication rule $*_3$ which operates between three gauge fields was also found.

The disk S-matrix elements of one closed and two open string states in the presence of an arbitrary background B-flux was evaluated in [10]. In the present paper, we would like to extend that calculation to the case of one closed string and an arbitrary number of open string states in the presence of a large background B-flux. From this calculation, we find multiplication rules $*_N$ between N open string fields. An interesting aspect of the new multiplication rule is that the difference between $*_N$ and $*$ is some total derivative terms. Hence, in the actions that involve both open and closed string fields, one must apply $*_N$ as the multiplication rule between open string fields. Whereas, in the tree level action that involves only the open string fields, like the non-commutative DBI action [5], this $*_N$ can be replaced with the $*$ product upon ignoring some total derivative terms. Moreover, in the one loop effective action of non-planer diagrams the $*_N$ does not operate between all the fields in the action, so the total derivative terms can not be ignored. Finally, it explains the appearance of $*$ product in the one loop effective action of planer diagrams in which $*_N$ operates between all the gauge fields in the action[11].

Another fact that the above S-matrix elements reveals is that the closed string fields in the non-commutative DBI action should be treated as functional of the non-commutative gauge fields. In particular, the S-matrix elements can be reproduced in field theory if one assumes that the closed string fields in this action are functional of the non-commutative

gauge field and then Taylor expands these fields around zero gauge field. Various terms in the Taylor expansion are proportional to the antisymmetric matrix θ^{ab} that appears in the definition of the $*$ product. Hence, this feature like the $*_N$ product is special to non-commutative field theories.

In [10], it was also shown that the result of disk S-matrix elements of one closed string NSNS state and two massless open string NS states can be reproduced in field theory by the ordinary DBI action and the SW map. One begins with expanding the DBI action around the background B-flux to produce an array of interactions. Then using the SW map, one transforms the array to non-commutative counterpart. The resulting non-commutative field theory reproduces exactly the above S-matrix elements. In particular, this calculation shows that the $*'$ product is reproduced in field theory by the SW map. In [15], it was shown that the $*_3$ product is reproduced by the SW map as well. We shall show that the various terms in the transformed DBI action that involve the antisymmetric matrix θ^{ab} can be combined properly to reproduce the appropriate terms in the Taylor expansion of the closed string fields. Hence, both $*_N$ product and the dependence of the closed string fields on non-commutative gauge fields seems to be reproduced in field theory by the ordinary DBI action and the SW map.

The remainder of the paper is organized as follows: We begin in section 2 by expanding the ordinary DBI action in Type 0 theories around the background B-flux, keeping the terms that involve one closed string tachyon and one, two or three massless open string fields. In Section 2.1 we transform these couplings between commutative fields to their non-commutative counterparts using the SW map. In particular, we combine the different terms that involve the matrix θ^{ab} in a compact form. This latter form is suggestive: the closed string fields in the DBI action should be treated as functional of the non-commutative gauge fields. In Section 3, we evaluate the disk S-matrix elements of one closed string tachyon and an arbitrary number of massless open string states in the presence of a large background B-flux. In particular, in this section we find the generalized star product $*_N$ in momentum space, and show that the difference between $*_N$ and $*$ is some total derivative terms. The S-matrix elements are fully consistent with the coupling found in Section 2.1 as expected. In Section 4, using the two features of the non-commutative field theories, *i.e.*, the dependence of closed string fields on the non-commutative gauge fields and the appearance of the $*_N$ product, we conjecture a non-commutative DBI action for transformation of the ordinary DBI action under the SW map.

2 Commutative DBI action

The world-volume theory of a single D-brane in type 0 theory includes a massless U(1) vector A_a and a set of massless scalars X^i , describing the transverse oscillations of the brane[16, 17]. The leading order low-energy effective action for the massless fields corresponds to

a dimensional reduction of a ten dimensional U(1) Yang Mills theory. As usual in string theory, there are higher order $\alpha' = \ell_s^2$ corrections, where ℓ_s is the string length scale. As long as derivatives of the field strengths (and second derivatives of the scalars) are small compared to ℓ_s , then the action takes a Dirac-Born-Infeld form [18]. To take into account the couplings of the massless open string states with closed strings, the DBI action may be extended to include massless Neveu-Schwarz closed string fields, *i.e.*, the metric, dilaton and Kalb-Ramond fields, and the closed string tachyon[19, 20]. In this way one arrives at the following world-volume action:

$$S = -T_p \int d^{p+1}\sigma g[T(\Phi)] e^{-\phi(\Phi)} \sqrt{-\det(P[G_{ab}(\Phi)] + B_{ab}(\Phi)) + \lambda F_{ab}} \quad (1)$$

where $\lambda = 2\pi\ell_s^2$, and the closed string tachyon function is $g[T] = 1 + T/4 + \dots$. Here, F_{ab} is the abelian field strength of the world-volume ordinary gauge field, while the metric and antisymmetric tensors are the pull-backs of the bulk tensors to the D-brane world-volume, *e.g.*,

$$P[G_{ab}] = G_{ab} + 2\lambda G_{i(a} \partial_{b)} \Phi^i + \lambda^2 G_{ij} \partial_a \Phi^i \partial_b \Phi^j \quad (2)$$

where we have used that fact that we are employing static gauge throughout the paper, *i.e.*, $\sigma^a = X^a$ for world-volume and $\lambda\Phi^i(\sigma^a)$ for transverse coordinates. In the action (1), we include in fact derivatives of the closed string fields by writing them as functional of the transverse scalar fields[21, 22], *e.g.*,

$$T(\Phi) = T^0 + \lambda\Phi^i \partial_i T^0 + \frac{\lambda^2}{2} \Phi^i \Phi^j \partial_i \partial_j T^0 + \frac{\lambda^3}{3!} \Phi^i \Phi^j \Phi^k \partial_i \partial_j \partial_k T^0 + \dots \quad (3)$$

To study the transformation of (1) under the SW map, we begin with expanding (1) for fluctuations around the background $G_{\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = \mathcal{F}^{ab} \eta_{a\mu} \eta_{b\nu}$, $\Phi = 0$ and $T = 0$ to extract the couplings expected from the DBI action. To simplify our illustration, we ignore all the closed string fields in (1) but the tachyon, that is,

$$S = -T_p \int d^{p+1}\sigma g[T(\Phi)] \sqrt{-\det(\eta_{ab} + \mathcal{F}_{ab} + \lambda F_{ab})} \quad (4)$$

Now it is straightforward, to expand eq. (4) using

$$\begin{aligned} \sqrt{\det(M_0 + M)} &= \sqrt{\det(M_0)} \left(1 + \frac{1}{2} \text{Tr}(M_0^{-1}M) - \frac{1}{4} \text{Tr}(M_0^{-1}MM_0^{-1}M) + \frac{1}{8} (\text{Tr}(M_0^{-1}M))^2 \right. \\ &\quad + \frac{1}{6} \text{Tr}(M_0^{-1}MM_0^{-1}MM_0^{-1}M) - \frac{1}{8} \text{Tr}(M_0^{-1}M) \text{Tr}(M_0^{-1}MM_0^{-1}M) \\ &\quad \left. + \frac{1}{48} (\text{Tr}(M_0^{-1}M))^3 + \dots \right) \end{aligned}$$

to produce a vast array of interactions. We are interested in the coupling linear in the tachyon field and linear, quadratic or triadic in massless open string fluctuations. The appropriate Lagrangian is:

$$\begin{aligned}
\mathcal{L}(T) &= -\frac{T_p c}{4} T(\Phi) \\
\mathcal{L}(T, A) &= -\frac{T_p \lambda c}{4} T(\Phi) \left(\frac{1}{2} \text{Tr}(VF) \right) \\
\mathcal{L}(T, 2A) &= -\frac{T_p \lambda^2 c}{4} T(\Phi) \left(-\frac{1}{4} \text{Tr}(VFVF) + \frac{1}{8} (\text{Tr}(VF))^2 \right) \\
\mathcal{L}(T, 3A) &= -\frac{T_p \lambda^3 c}{4} T(\Phi) \left(\frac{1}{6} \text{Tr}(VFVFVF) - \frac{1}{8} \text{Tr}(VF) \text{Tr}(VFVF) + \frac{1}{48} (\text{Tr}(VF))^3 \right)
\end{aligned} \tag{5}$$

In the above Lagrangian,

$$c \equiv \sqrt{-\det(\eta_{ab} + \mathcal{F}_{ab})} \quad , \quad V^{ab} \equiv ((\eta + \mathcal{F})^{-1})^{ab} \tag{6}$$

Using the Taylor expansion (3), one includes the scalar fields in the above couplings. Now we transform these couplings to their non-commutative counterparts under the SW map.

2.1 Change of variables

In [5] differential equation for transforming ordinary gauge field to its non-commutative counterpart was found to be

$$\begin{aligned}
\delta \hat{A}_a(\theta) &= \frac{1}{4} \delta \theta^{cd} \left(\hat{A}_c * \hat{F}_{ad} + \hat{F}_{ad} * A_c - \hat{A}_c * \partial_d \hat{A}_a - \partial_d \hat{A}_a * A_c \right) \\
\delta \hat{F}_{ab}(\theta) &= \frac{1}{4} \delta \theta^{cd} \left(2 \hat{F}_{ac} * \hat{F}_{bd} + 2 \hat{F}_{bd} * \hat{F}_{ac} - \hat{A}_c * (\hat{D}_d \hat{F}_{ab} + \partial_d \hat{F}_{ab}) - (\hat{D}_d \hat{F}_{ab} + \partial_d \hat{F}_{ab}) * \hat{A}_c \right)
\end{aligned} \tag{7}$$

where the gauge field strength and $*$ product were defined to be

$$\begin{aligned}
\hat{F}_{ab} &= \partial_a \hat{A}_b - \partial_b \hat{A}_a - i \hat{A}_a * \hat{A}_b + i \hat{A}_b * \hat{A}_a \\
&= \partial_a \hat{A}_b - \partial_b \hat{A}_a - i [\hat{A}_a, \hat{A}_b]_M \\
f(x) * g(x) &= e^{i \partial_1 \Lambda \partial_2} f(x_1) g(x_2) |_{x_1=x} ,
\end{aligned}$$

where

$$\partial_1 \Lambda \partial_2 = \frac{1}{2} \theta^{ab} \frac{\partial}{\partial x_1^a} \frac{\partial}{\partial x_2^b} \tag{8}$$

These differential equations were solved in [10] to order $O(\hat{A}^2)$ by integration along a special path in the space of the matrix θ^{ab} . It gives a relation between ordinary fields appearing in

(5) and non-commutative fields corresponding to open string vertex operators. The result for abelian case that we are interested in is

$$\begin{aligned} A_a &= \hat{A}_a + \frac{1}{2}\theta^{bc}\hat{A}_b *' (\hat{F}_{ca} + \partial_c\hat{A}_a) + O(\hat{A}^3) \\ F_{ab} &= \hat{F}_{ab} - \theta^{cd} (\hat{F}_{ac} *' \hat{F}_{bd} - \hat{A}_c *' \partial_d\hat{F}_{ab}) + O(\hat{A}^3) \end{aligned} \quad (9)$$

where the multiplication rule between two fields operates as

$$f(x) *' g(x) = \frac{\sin(\partial_1\Lambda\partial_2)}{\partial_1\Lambda\partial_2} f(x_1)g(x_2)|_{x_i=x} .$$

These solutions were improved in [15] to include the terms of order $O(\hat{A}^3)$, that is,

$$\begin{aligned} A_a &= \hat{A}_a + \frac{1}{2}\theta^{bc}\hat{A}_b *' (\partial_c\hat{A}_a + \hat{F}_{ca}) \\ &\quad + \frac{1}{2}\theta^{bc}\theta^{de} (-\hat{A}_b\partial_d\hat{A}_a(\partial_c\hat{A}_e + \hat{F}_{ce}) + \partial_b\partial_d\hat{A}_a\hat{A}_c\hat{A}_e + 2\partial_d\hat{A}_b\partial_a\hat{A}_c\hat{A}_e) *_3 + O(\hat{A}^4) \\ F_{ab} &= \hat{F}_{ab} - \theta^{cd} (\hat{F}_{ac} *' \hat{F}_{bd} - \hat{A}_c *' \partial_d\hat{F}_{ab}) \\ &\quad + \frac{1}{2}\theta^{cd}\theta^{ef} (\partial_c\partial_e(\hat{F}_{ab}\hat{A}_d\hat{A}_f) - \partial_e(\hat{F}_{cd}\hat{F}_{ab}\hat{A}_f) + 2\partial_e(\hat{F}_{ac}\hat{F}_{bd}\hat{A}_f) \\ &\quad - (\hat{F}_{ac}\hat{F}_{bd}\hat{F}_{ef}) + \frac{1}{4}(\hat{F}_{ab}\hat{F}_{cd}\hat{F}_{ef}) + \frac{1}{2}(\hat{F}_{ab}\hat{F}_{de}\hat{F}_{cf}) - 2(\hat{F}_{ce}\hat{F}_{af}\hat{F}_{bd})) *_3 + O(\hat{A}^4) \end{aligned} \quad (10)$$

where the multiplication rules between three fields is

$$(f(x)g(x)h(x)) *_3 = \left(\frac{\sin(\partial_2\Lambda\partial_3)}{(\partial_1 + \partial_2)\Lambda\partial_3} \frac{\sin(\partial_1\Lambda(\partial_2 + \partial_3))}{\partial_1\Lambda(\partial_2 + \partial_3)} + 1 \leftrightarrow 2 \right) f(x_1)g(x_2)h(x_3)|_{x_i=x}$$

The eq. (10) may be used to find the transformation of the scalar fields in adjoint representation of $U(1)$, for example,

$$\Phi_i = \hat{\Phi}_i + \theta^{bc}\hat{A}_b *' D_c\hat{\Phi}_i + \frac{1}{2}\theta^{bc}\theta^{de} (-\hat{A}_b\partial_d\hat{\Phi}_i(\partial_c\hat{A}_e + \hat{F}_{ce}) + \partial_b\partial_d\hat{\Phi}_i\hat{A}_c\hat{A}_e) *_3 + \dots \quad (11)$$

where dots represent terms which involve more than three open string fields. They produce couplings between more than three fields upon replacing them into (5) in which we are not interested in this section. In above equation $D_c\hat{\Phi}_i = \partial_c\hat{\Phi}_i - i[\hat{A}_c, \hat{\Phi}_i]_M$.

The differential equation (7) expresses infinitesimal variation of linear field, *e.g.*, $\delta\hat{A}_a$, in terms of infinitesimal variation of the non-commutative parameters, *i.e.*, $\delta\theta^{cd}$. Upon integration (10), this transforms the linear commutative gauge field in terms of nonlinear combination of non-commutative fields. However, we are interested in transforming nonlinear combinations of commutative fields appearing in (5) in terms of non-commutative fields. Such a transformation, in principle, might be found from a differential equation alike

(7) that expresses infinitesimal variation of the non-linear fields in terms of infinitesimal variation of non-commutative parameter. Upon integration, that would produce the desired transformation. In that way, one would find that not only the fields transform as in (10) but the multiplication rule between fields also undergo appropriate transformation. We are not going to find such a differential equations here. Instead, we simply note that the transformation for multiplication rule between two and three open string fields can be read from the right hand side of eq. (10) to be

$$\begin{aligned} fg|_{\theta=0} &\longrightarrow f *' g|_{\theta \neq 0} \\ fgh|_{\theta=0} &\longrightarrow (fgh) *_3|_{\theta \neq 0} \end{aligned} \quad (12)$$

for f, g and h being any arbitrary open string fields.

Now with the help of equation (10), (11) and (12), one can transform the commutative Lagrangian (5) to non-commutative counterpart. In doing so, one should first ignore the multiplication rules in (10) and (11) *i.e.*, $*'$ and $*_3$. Then using the resulting transformations, one maps the commutative couplings in (5) to their non-commutative counterparts. Finally, one should replace the generalized star product between the non-commutative fields $\hat{\Phi}^i$, \hat{A}_a or \hat{F}_{ab} ¹. Let us work more details on the coupling between the tachyon, the transverse scalar and the gauge field. One starts with the following couplings

$$-\frac{T_p c}{4} \left(\lambda \partial_i T^0 \Phi^i + \lambda^2 \partial_i T^0 \Phi^i V_A^{ab} \partial_b A_a \right)$$

which is the first Taylor expansion of $\mathcal{L}(T) + \mathcal{L}(T, A)$. Now one transforms the commutative fields in the above couplings to their non-commutative fields using the transformations (10) and (11). Since we are interested in the terms with two open string fields, one should replace the open string fields in the second term above with their non-commutative fields, *i.e.*, first terms of (10) and (11). Whereas, in the first term the scalar field must be replaced with the second term of (11). In this way one finds²,

$$-\frac{T_p \lambda^2 c}{4} \left(\partial_i T^0 V_A^{ab} \hat{A}_a \partial_b \hat{\Phi}^i + \partial_i T^0 \hat{\Phi}^i V_A^{ab} \partial_b \hat{A}_a \right)$$

Finally, one should insert the multiplication rule between the two open string fields, that is

$$\begin{aligned} \hat{\mathcal{L}}(T^0, \hat{\Phi}, \hat{A}) &= -\frac{T_p \lambda^2 c}{4} \partial_i \partial_a T^0 (\hat{\Phi}^i *' \hat{A}^a) \\ \hat{\mathcal{L}}(T^0, 2\hat{A}) &= -\frac{T_p \lambda^2 c}{4} \left(\frac{1}{2} \partial_a \partial_b T^0 (\hat{A}^a *' \hat{A}^b) - \frac{1}{4} T^0 (V_S^{ab} \hat{F}_{bc} V_S^{cd} *' \hat{F}_{da}) \right) \\ \hat{\mathcal{L}}(T^0, 2\hat{\Phi}, \hat{A}) &= -\frac{T_p \lambda^3 c}{8} \partial_a \partial_i \partial_j T^0 (\hat{\Phi}^i \hat{\Phi}^j \hat{A}^a) *_3 \end{aligned} \quad (13)$$

¹Note that $*'$ and $*_3$ products are invariant under all permutations of the non-commutative fields[15].

²Note that our conventions set $\theta^{ab} = \lambda(V_A)^{ab}$ where V_A is the antisymmetric part of the V-matrix (6).

$$\begin{aligned}\hat{\mathcal{L}}(T^0, \hat{\Phi}, 2\hat{A}) &= -\frac{T_p \lambda^3 c}{4} \left(\frac{1}{2} \partial_a \partial_b \partial_i T^0 (\hat{\Phi}^i \hat{A}^a \hat{A}^b) *_3 - \frac{1}{4} \partial_i T^0 (\hat{\Phi}^i V_S^{ab} \hat{F}_{bc} V_S^{cd} \hat{F}_{da}) *_3 \right) \\ \hat{\mathcal{L}}(T^0, 3\hat{A}) &= -\frac{T_p \lambda^3 c}{4} \left(\frac{1}{3!} \partial_a \partial_b \partial_c T^0 (\hat{A}^a \hat{A}^b \hat{A}^c) *_3 - \frac{1}{4} \partial_e T^0 (\hat{A}^e V_S^{ab} \hat{F}_{bc} V_S^{cd} \hat{F}_{da}) *_3 \right)\end{aligned}$$

where $\hat{A}^a = V_A^{ab} \hat{A}_b$, and $V_A(V_S)$ is the antisymmetric(symmetric) part of the V -matrix (6). We have also listed in the above equation the results for some other couplings and ignored some total derivative terms. In the above equations, the fields inside the argument of $\hat{\mathcal{L}}(\dots)$ refers to the number of fields in the first term of each equation.

Apart from the multiplication rules, V_A^{ab} appears only in the first terms in (13). These terms looks like the Taylor expansion (3), but for both $\hat{\Phi}^i$ and \hat{A}^a . The appearance of V_S^{ab} in the second term is consistent with the fact that the metric for the open string fields is V_S^{ab} [5]. Now we turn to the string theory side to illustrate that the above terms are produced by the string theory S-matrix elements, and then generalize above terms to N open string fields.

3 Scattering Calculations

In this Section we evaluate the coupling of one closed string tachyon with an arbitrary number of massless open string states on the world-volume of D-brane with large background B-flux. The corresponding scattering amplitude is given as

$$A_{12\dots N} = \frac{\lambda^N T_p c}{2(2\pi)^{N-1}} \int dx_1 \cdots dx_N d^2 z \langle 0 | T \left(V_1^{NS} \cdots V_N^{NS} V^T \right) | 0 \rangle , \quad (14)$$

we have normalized the amplitude such that it reproduces the numerical factors in (13). The vertex operators above are

$$\begin{aligned}V_\ell^{NS}(k_\ell, \zeta_\ell, x_\ell) &= (\zeta_\ell \cdot \mathcal{G})_\mu : V_0^\mu(2k_\ell \cdot V^T, x_\ell) : \quad \ell = 1, 2, \dots, N \\ V^T(p, z, \bar{z}) &= : V_{-1}(p, z) : : V_{-1}(p \cdot D, \bar{z}) : ,\end{aligned}$$

where the momenta and polarizations satisfy $p^2 = 1$, $k_\ell \cdot V_S \cdot k_\ell = 0$ and $k_\ell \cdot V_S \cdot \zeta_\ell = 0$. The V_0^μ and V_{-1} are given as

$$\begin{aligned}V_0^\mu(p, z) &= (\partial X^\mu + ip \cdot \psi \psi^\mu) e^{ip \cdot X} \\ V_{-1}(p, z) &= e^{-\sigma} e^{ip \cdot X}\end{aligned} \quad (15)$$

The vertex operators above are chosen such that they saturate the background super-ghost charge on the world-sheet, *i.e.*, $Q_\sigma = 2$. In the open string vertex operators, the index μ will run over the world-volume (transverse) directions when it represents a world-volume vector (transverse scalar) state. Here we are using the notation of ref. [23]. In particular,

we have used the doubling trick [25, 24] to convert the disk amplitude to a calculation involving only the standard holomorphic correlators

$$\begin{aligned}\langle 0|T(X^\mu(z_1)X^\nu(z_2))|0\rangle &= -\eta^{\mu\nu}\log(z_1-z_2) - \frac{i\pi}{2}\mathcal{F}^{\mu\nu}\Theta(x_1-x_2) \\ \langle 0|T(\psi^\mu(z_1)\psi^\nu(z_2))|0\rangle &= -\frac{\eta^{\mu\nu}}{z_1-z_2} \\ \langle 0|T(\sigma(z_1)\sigma(z_2))|0\rangle &= -\log(z_1-z_2).\end{aligned}\tag{16}$$

where the second term in the right hand side of the first equation above is nonzero when both z_1 and z_2 are on the real axis (the boundary of the world-sheet), and $\Theta(x) = 1(-1)$ if $x > 0(x < 0)$. For more details on our conventions, we refer the interested reader to the Appendix of [10]. Replacing the vertex operators into the scattering amplitude (14), one finds

$$\begin{aligned}A_{12\dots N} &= \frac{\lambda^N T_p c}{2(2\pi)^{N-1}} (\zeta_1 \cdot \mathcal{G})_{\alpha_1} \cdots (\zeta_N \cdot \mathcal{G})_{\alpha_N} \int dx_1 \cdots dx_N d^2 z \\ &\quad \langle 0|T\left(:(\partial X^{\alpha_1}(x_1) + 2ik_1 \cdot V^T \cdot \psi(x_1) \psi^{\alpha_1}(x_1))e^{2ik_1 V^T X(x_1)}:\right. \\ &\quad \left.:(\partial X^{\alpha_N}(x_N) + 2ik_N \cdot V^T \cdot \psi(x_N) \psi^{\alpha_N}(x_N))e^{2ik_N V^T X(x_N)}:\right. \\ &\quad \left.:e^{-\phi(z)}e^{ipX(z)}: e^{-\phi(\bar{z})}e^{ipDX(\bar{z})}:\right)|0\rangle\end{aligned}\tag{17}$$

Using the Wick theorem, one can evaluate the correlation function above with the help of the two-point functions (16). When α_n takes value in the transverse space, in order to produce the Taylor expansion of the closed string field in terms of non-commutative scalar fields, one must chose only the ∂X^{α_n} of the corresponding open string vertex operator, and it should contracts only with the closed string vertex operator. That is,

$$\frac{ip^{\alpha_n}}{(x_n - z)} + \frac{i(p \cdot D)^{\alpha_n}}{(x_n - \bar{z})} = \frac{i(z - \bar{z})p^{\alpha_n}}{(x_n - z)(x_n - \bar{z})}$$

where we have used the identity $(p \cdot D)^{\alpha_n} + p^{\alpha_n} = 0$.

When α_m takes value in the world-volume space, in general, the amplitude (17) contains many poles and contact terms. However, for the special case that the V-matrix is antisymmetric, *i.e.*, large background B-flux, the amplitude simplifies considerably. First we consider the correlation between the world-sheet fermions. The contractions between ψ 's yield different terms which are proportional to one of the following terms:

$$\zeta_i \cdot \mathcal{G} \cdot V \cdot \zeta_i, \quad \zeta_i \cdot \mathcal{G} \cdot V \cdot k_j, \quad k_i \cdot \mathcal{G} \cdot V \cdot k_j.$$

However, using the identity $\mathcal{G} \cdot V = V_S$ (see the Appendix in [10]), these terms do not appear for the case that we are interested in, *e.g.*, V_A . Next we consider the ∂X^{α_m} part of the open string vertex operators. Contractions of these parts among the open string vertex operators

result terms of the above form which have zero effect. Whereas, their contractions with the closed string vertex operator yield the following terms:

$$\frac{ip^{\alpha_m}}{(x_m - z)} + \frac{i(p \cdot D)^{\alpha_m}}{(x_m - \bar{z})} = \frac{i(z - \bar{z})p^{\alpha_m}}{(x_m - z)(x_m - \bar{z})}$$

where we have used the conservation of momentum in the world-volume directions

$$\sum_{\ell=1}^N 2k_{\ell} \cdot V^T + p + p \cdot D = 0$$

and used the fact that the first term has zero result when V-matrix is antisymmetric. Finally, it is straightforward exercise to evaluate the correlations between the exponential operators in (17) using the two-point functions in (16). The final result is,

$$\begin{aligned} A_{12\dots N} &= \frac{\lambda^N T_p c}{2(2\pi)^{N-1}} \prod_{\ell=1}^N (\zeta_{\ell} \cdot \mathcal{G} \cdot p) \int dx_1 \dots dx_N d^2 z \left(\prod_{i < j} e^{-i\pi l_{ij} \Theta(x_i - x_j)} \right) \\ &\quad \times i^N (z - \bar{z})^{N-2} \prod_{i=1}^N (x_i - z)^{-1 + \sum_{j=1}^{j=N} l_{ij}} (x_i - \bar{z})^{-1 - \sum_{j=1}^{j=N} l_{ij}} \end{aligned} \quad (18)$$

where $l_{ij} = 2k_i \cdot V_A \cdot k_j$. A nontrivial check of the result in eq. (18) is that the integrals is $SL(2, R)$ invariant. To remove the associated divergence and properly evaluate the amplitude, we fix: $z = i$, $\bar{z} = -i$, $x_1 = R \rightarrow \infty$. With this choice, one finds

$$\begin{aligned} A_{12\dots N} &= \frac{\lambda^N T_p c}{4\pi^{N-1}} \prod_{\ell=1}^N (\zeta_{\ell} \cdot \mathcal{G} \cdot p) \int_{-\infty}^{+\infty} dx_2 \int_{x_2}^{+\infty} dx_3 \dots \int_{x_{N-1}}^{+\infty} dx_N \\ &\quad \left(\prod_{i < j} e^{-i\pi l_{ij} \Theta(x_i - x_j)} \right) \prod_{i=2}^N (x_i - i)^{-1 + \sum_{j=1}^{j=N} l_{ij}} (x_i + i)^{-1 - \sum_{j=1}^{j=N} l_{ij}} \end{aligned} \quad (19)$$

It is not difficult to check that the amplitude above has no pole. Hence, this amplitude defines the *coupling* between one closed string tachyon and N massless open string fields when the background B-flux is large. The integral in the amplitude is then related to the multiplication rule between the N external open string fields.

The amplitude (19) is one ordering of the external open string states, *i.e.*, $(23 \dots N1)$. Adding all non-cyclic permutation of the external open string states, one finds the full coupling between the tachyon and N massless open string fields in the momentum space,

$$A = \frac{\lambda^N T_p c}{4n!m!} \left(\prod_{\ell=1}^N (\zeta_{\ell} \cdot \mathcal{G} \cdot p) \right) *_N \quad (20)$$

where we have also divided the amplitude by $n!m!$, where n and m are the number of open string scalar and gauge fields, respectively. The multiplication rule between the N open

string fields in the momentum space is then

$$(f_1 \cdots f_N) *_N = \frac{1}{\pi^{N-1}} \sum \int_{-\infty}^{+\infty} dx_2 \int_{x_2}^{+\infty} dx_3 \cdots \int_{x_{N-1}}^{+\infty} dx_N \left(\prod_{i < j} e^{-i\pi l_{ij} \Theta(x_i - x_j)} \right) \prod_{i=2}^N (x_i - i)^{-1 + \sum_{j=1}^{j=N} l_{ij}} (x_i + i)^{-1 - \sum_{j=1}^{j=N} l_{ij}} \quad (21)$$

where the summation is over all permutations of $2, 3, \dots, N$, *e.g.*, non-cyclic permutation of $1, 2, \dots, N$. An alternative formula for the $*_N$ was also found in [11]. When the background B-flux vanishes, the above multiplication rule reduces to ordinary product between N fields, that is

$$\lim_{l_{ij} \rightarrow 0} (f_1 \cdots f_N) *_N = 1 \quad (22)$$

It should not be hard to perform the integrals in (21) for any N , however, we perform the calculations for $N = 2, 3$. The results are

$$(f_1 f_2) *_2 = \frac{\sin(\pi l_{12})}{\pi l_{12}} = f_1 *' f_2 \quad (23)$$

$$(f_1 f_2 f_3) *_3 = \frac{-1}{4\pi^2} \sum \left(\frac{e^{i\pi(l_{21} + l_{31} + l_{23})}}{(l_{12} + l_{13})(l_{12} + l_{32})} + \frac{e^{i\pi(l_{12} + l_{13} + l_{23})}}{(l_{31} + l_{32})(l_{31} + l_{21})} + \frac{e^{i\pi(l_{12} + l_{32} + l_{31})}}{(l_{31} + l_{32})(l_{12} + l_{32})} \right)$$

for $N = 3$, the double integral is evaluated in the Appendix. This momentum representation of the $*_3$ is exactly the one found in [11] in studying the one loop effective action of non-planer diagrams.

When the closed string tachyon has no momentum in the world-volume directions, *i.e.*, $p^a = 0$, one expects that the $*_N$ product would reduce to the familiar $*$ product. This can be seen from (21) by noting that when p^a is zero $\sum_{j=1}^{j=N} l_{ij} = 0$ for any i . This follows from the conservation of momentum in the world-volume directions, *i.e.*, $\sum_\ell k_\ell^a + p^a = 0$. Therefore,

$$(f_1 \cdots f_N) *_N = \left(\prod_{i < j} e^{-i\pi l_{ij} \Theta(x_i - x_j)} \right) \times \frac{1}{\pi^{N-1}} \sum \int_{-\infty}^{+\infty} dx_2 \int_{x_2}^{+\infty} dx_3 \cdots \int_{x_{N-1}}^{+\infty} dx_N \prod_{i=2}^N (x_i^2 + 1)^{-1} = \prod_{i < j} e^{-i\pi l_{ij} \Theta(x_i - x_j)} = f_1 * f_2 * \cdots * f_N \quad (24)$$

where in the first line above we have used the fact that the factor

$$\prod_{i < j} e^{-i\pi l_{ij} \Theta(x_i - x_j)}$$

depends only on the cyclic ordering of $1, 2, 3, \dots, N$ [5]. This shows that if one ignores total derivative terms, *i.e.*, p^a , the $*_N$ product reduces to the $*$ product between N open string fields. While it was simple exercise using eq. (21), to show the difference between $*_N$ and $*$ product is some total derivative terms in the world volume directions, it is nontrivial calculation to show this fact using the integrated form of the $*_N$ as in (23).

Using the fact that $\mathcal{G}^{ij} = N_{ij}$ and $\mathcal{G}_A^{ab} = -V_A^{ab}$ (see Appendix of [10]), it is not difficult to check that the first terms in (13) are exactly reproduced by the couplings in (20).

4 Non-commutative DBI action

The string theory calculations (20) shows that the coupling between the closed string tachyon and N massless open string states when $V_A \gg V_S$ is consistent with the following Lagrangian in field theory:

$$\hat{\mathcal{L}}(T) = -\frac{T_p c}{4} T(\hat{\Phi}, \hat{A}) *_N \quad (25)$$

where

$$\begin{aligned} T(\hat{\Phi}, \hat{A}) &= \sum_{n=0, m=0}^{\infty} \frac{\lambda^{n+m}}{n! m!} \left(\hat{\Phi}^{i_1} \dots \hat{\Phi}^{i_n} \hat{A}^{a_1} \dots \hat{A}^{a_m} \right) \\ &\quad (\partial_{x^{i_1}} \dots \partial_{x^{i_n}}) (\partial_{x^{a_1}} \dots \partial_{x^{a_m}}) T^0(x^a, x^i) |_{x^i=0, x^a=\sigma^a} \end{aligned} \quad (26)$$

and $\hat{A}^a = V_A^{ab} \hat{A}_b$, and the $*_N$ operates as the multiplication rules between the N open string fields. If we were able to evaluate the string S-matrix element (17) for arbitrary background B-flux, then we could find the couplings between the tachyon and the N open string states that involve both V_A and V_S . The result should reproduce with couplings like those in (13). The Lagrangian (25) and the couplings in (13) are consistent with the following action:

$$\hat{S} = -\frac{T_p c}{\sqrt{-\det(V_S)}} \int d^{p+1} \sigma \left(g[T(\hat{\Phi}, \hat{A})] \sqrt{-\det((V_S)_{ab} + \lambda \hat{F}_{ab})} \right) *_N \quad (27)$$

where the $*_N$ is the multiplication rule between \hat{F}_{ab} and the open string fields stemming from the Taylor expansion (26). This action is of course consistent with the non-commutative DBI action [5] in which there is no closed string field. In that case, the $*_N$ operates as the multiplication rule between all the fields in the action, hence, using (24) it can be replaced by the $*$ product upon ignoring some total derivative terms.

The action (27) can still be extended further to include the massless closed string fields as well. If one extracts the coupling between one graviton and one or two gauge fields

from the ordinary DBI action (1), and then uses the SW map (10) to transform them to non-commutative counterparts, one will find the following terms:

$$\begin{aligned}\hat{\mathcal{L}}(h^0, \hat{A}) &= -\frac{1}{2}T_p c \left(\hat{A}^a \text{Tr}(V \partial_a h^0) - \text{Tr}(V h^0 V \hat{F}) \right) \\ \hat{\mathcal{L}}(h^0, 2\hat{A}) &= -\frac{1}{2}T_p c \left(\frac{1}{2} \hat{A}^a *' \hat{A}^b \text{Tr}(V \partial_a \partial_b h^0) - \frac{1}{2} \text{Tr}(V h^0) \text{Tr}(V_S \hat{F} *' V_S \hat{F}) \right. \\ &\quad \left. - \hat{A}^a *' \text{Tr}(V \partial_a h^0 V \hat{F}) + \text{Tr}(V h^0 V \hat{F} *' V_S \hat{F}) \right)\end{aligned}\quad (28)$$

Note that the V_S^{ab} , as in (13), appears only when both indices of V^{ab} contract with the open string fields. This is consistent with the fact that the V_S is the open string metric[5]. The above terms are consistent with the following prescribed action

$$\begin{aligned}\hat{S} &= -T_p \int d^{p+1}\sigma \\ &\quad \left(g[T(\hat{\Phi}, \hat{A})] e^{-\phi(\hat{\Phi}, \hat{A})} \sqrt{-\det(P_\theta[G_{ab}(\hat{\Phi}, \hat{A}) + B_{ab}(\hat{\Phi}, \hat{A})] + \lambda \hat{F}_{ab})} \right) *_N\end{aligned}\quad (29)$$

where now the definition of the pull-back P_θ is the extension of (2) in which ordinary derivative is replaced by its non-commutative covariant derivative, *i.e.*, $\partial_a \Phi^i \longleftrightarrow D_a \hat{\Phi}^i = \partial_a \hat{\Phi}^i - i[\hat{A}_a, \hat{\Phi}^i]_M$. Expanding above action around the background B-flux, one finds various couplings between open and closed string fields. When both indices of V^{ab} contract with the open string fields one must replace it with V_S^{ab} . The resulting terms are then fully consistent with the terms in (28). When there is no background B-flux, the above non-commutative action reduces to the ordinary DBI action (1). It is also consistent with the non-commutative DBI action in which no closed string field is included [5]. Hence it seems reasonable to believe that the above action is the transformation of the ordinary DBI action (1) under the SW map.

The appearance of the $*_N$ and the dependence of the closed string fields on the non-commutative gauge fields seems to indicate that the non-commutative theory (29) is not gauge invariant. However, the S-matrix elements calculated in this theory satisfy the Ward identity³. Unlike the commutative theory (1) that contact terms and poles of the scattering amplitudes are separately satisfy the Ward identity, in the non-commutative case (29), however the combination of contact terms and the poles of scattering amplitude satisfy the Ward identity[10].

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³The scattering amplitudes are zero under replacing polarization of the external states with their momenta.

A The $*_3$ evaluation

In this appendix, we evaluate the double integral that appears in the $*_3$ product (21), that is

$$(f_1 f_2 f_3) *_3 = \frac{1}{\pi^2} \sum e^{-i\pi(l_{12}+l_{13}-l_{23})} \int_{-\infty}^{+\infty} dx_2 \int_{x_2}^{+\infty} dx_3 \\ \times (x_2 - i)^{-1+l_{21}+l_{23}} (x_2 + i)^{-1-l_{21}-l_{23}} (x_3 - i)^{-1+l_{31}+l_{32}} (x_3 + i)^{-1-l_{31}-l_{32}}$$

In the Appendix of [26], the following basic integral is evaluated:

$$L \equiv (2i)^{(-3-a-b-c-d-e)} \int_{-\infty}^{+\infty} dx_2 \int_{x_2}^{+\infty} dx_3 (x_2 - i)^a (x_2 + i)^b (x_3 - i)^c (x_3 + i)^d (x_3 - x_2)^e \\ = -\Gamma(-2 - a - b - c - d - e) \left\{ (-i)^{2(a+c)} \sin[\pi(b + d + e)] \right. \\ \times \frac{\Gamma(-1 - d - e) \Gamma(1 + e) \Gamma(2 + b + d + e)}{\Gamma(-d) \Gamma(-a - c)} \\ \times {}_3F_2(-c, 1 + e, 2 + b + d + e; 2 + d + e, -a - c; 1) \\ + (-i)^{2(a+c+d+e)} \sin(\pi b) \frac{\Gamma(1 + d + e) \Gamma(1 + b) \Gamma(-1 - c - d - e)}{\Gamma(-c) \Gamma(-1 - a - c - d - e)} \\ \left. \times {}_3F_2(-d, -1 - c - d - e, 1 + b; -d - e, -1 - a - c - d - e; 1) \right\} .$$

Using this result, one finds

$$(f_1 f_2 f_3) *_3 = -\frac{i}{2\pi^2} \sum \left(e^{i\pi l_{23}} \sin[\pi(l_{21} + l_{31})] \right. \\ \times \frac{\Gamma(l_{31} + l_{32}) \Gamma(-l_{21} - l_{31})}{\Gamma(1 + l_{31} + l_{32}) \Gamma(2 - l_{31} - l_{21})} {}_2F_1(1, -l_{21} - l_{31}; 2 - l_{31} - l_{21}; 1) \\ + e^{-i\pi l_{13}} \sin[\pi(l_{21} + l_{23})] \\ \left. \times \frac{\Gamma(-l_{31} - l_{32}) \Gamma(-l_{21} - l_{23})}{\Gamma(1 - l_{31} - l_{32}) \Gamma(2 - l_{21} - l_{23})} {}_2F_1(1, -l_{21} - l_{23}; 2 - l_{21} - l_{23}; 1) \right)$$

where we have used the fact that ${}_3F_2(a, b, c; a, d; 1) = {}_2F_1(b, c; d; 1)$. Now using the identity ${}_2F_1(1, a; 2 + a; 1) = 1 + a$, and the properties of the Gamma functions, one finds

$$(f_1 f_2 f_3) *_3 = \frac{1}{4\pi^2} \sum \left(\frac{e^{i\pi(l_{21} + l_{31} + l_{23})} - e^{-i\pi(l_{21} + l_{31} - l_{23})}}{(l_{31} + l_{32})(l_{31} + l_{21})} + \frac{e^{i\pi(l_{21} + l_{23} - l_{13})} - e^{-i\pi(l_{21} + l_{23} + l_{13})}}{(l_{31} + l_{32})(l_{12} + l_{32})} \right)$$

after some rearranging the right hand side, one finds the result in (23).

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